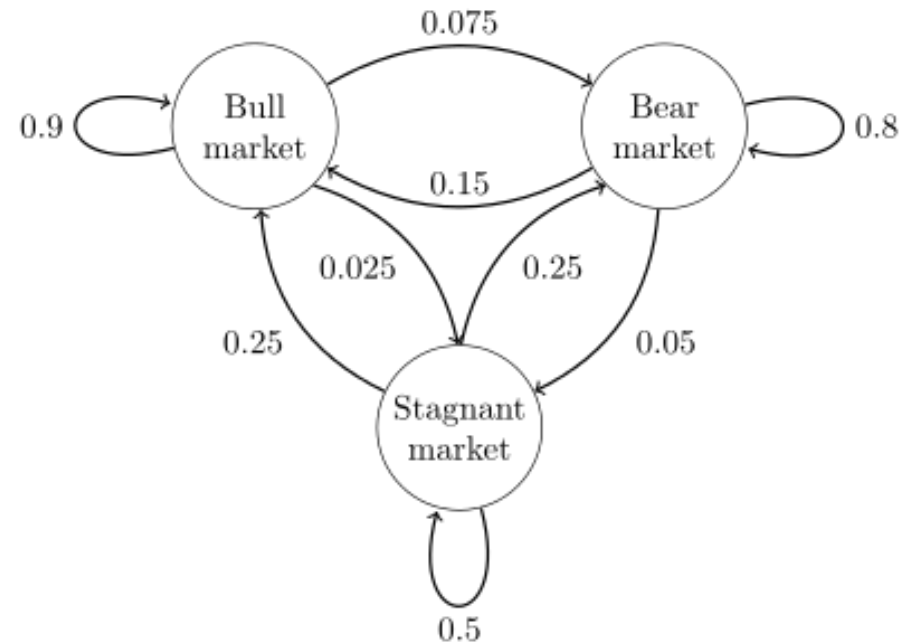


CS 6511: Artificial Intelligence

Markov Models

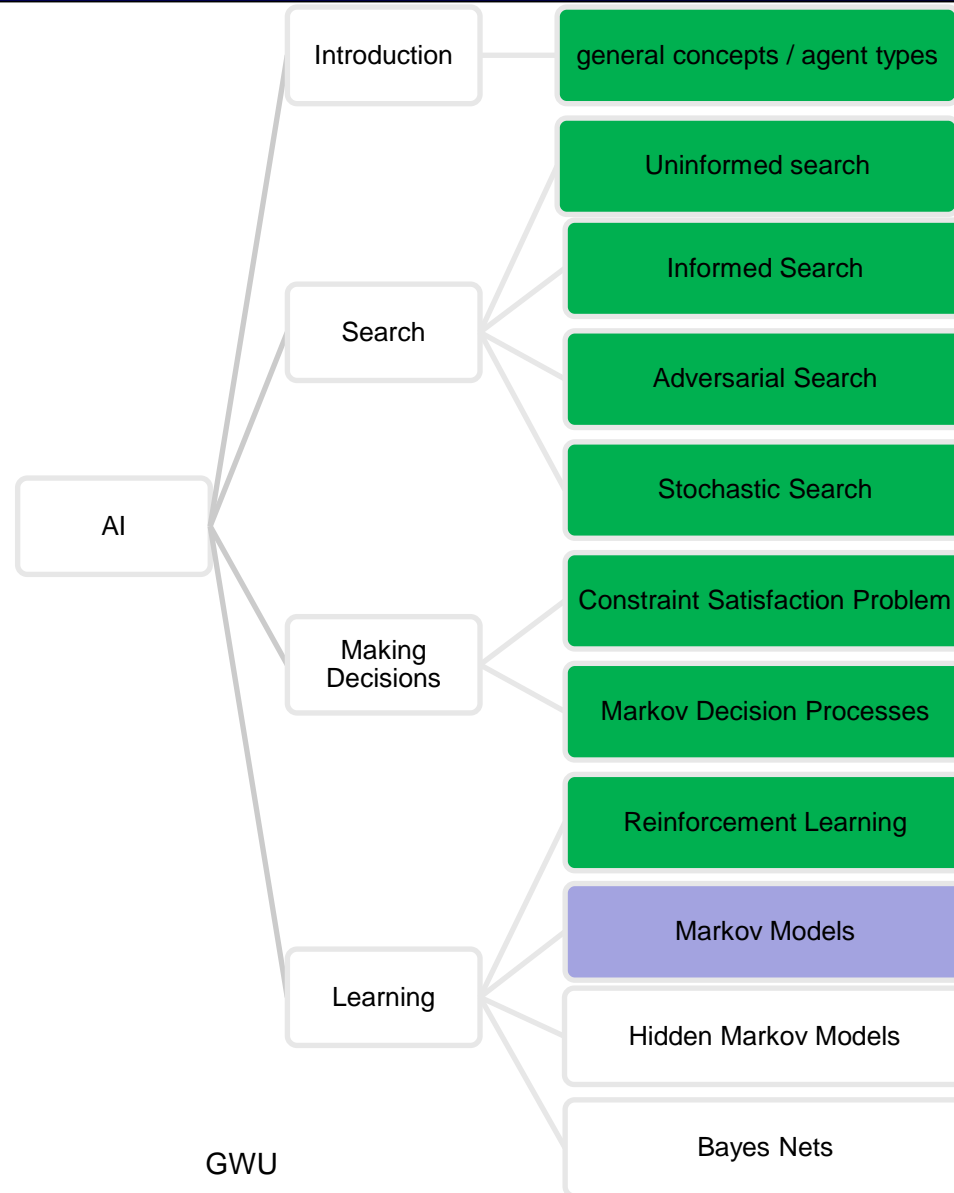


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[Original version of these slides was created by Dan Klein and Pieter Abbeel for Intro to AI at UC Berkeley. <http://ai.berkeley.edu>]

Outline



Agenda for Today

- Bit more probability
 - Independence
 - Conditional Independence
- Markov Models
- Introduce Hidden Markov Models

Many Applications, in Many Areas

■ Applications

- Speech recognition
- Robot localization
- User attention
- Shape identification
- Medical monitoring

■ Areas

- Medicine
- Genetics
- eCommerce
- Stock Market
- Natural Language Processing
- Battlefield/Defense

But First, a Quiz

- Context: A video game, called “AI_Among_Us”
- There are 90% villagers and 10% imposters in the game
- Imposters wear orange and blue with 50% probability each.
- Villagers always wear blue.
- You see a person – they are wearing blue.
- What is the probability it is an imposter?

- $P(m/b) = P(b/m) P(m) / P(b)$
 $= 0.5 \cdot 0.1 / (P(b/m) P(m) + P(b/v) P(v))$
 $= 0.5 * 0.1 / (0.5 * 0.1 + 1 * 0.9)$
 $= 0.05 / (0.05 + 0.9)$
 $= 0.05 / 0.95$
 $= 0.053$

Imagine 1000 people → 900 V, 100 I. 900 blue villager, 50 blue imposter. See blue. Probability of Imposter = $50 / 950 = 0.053$

Independence

- Two variables are *independent* in a joint distribution if:

$$P(X, Y) = P(X)P(Y)$$

$$X \perp\!\!\!\perp Y$$

$$\forall x, y P(x, y) = P(x)P(y)$$

- Says the joint distribution *factors* into a product of two simple ones
 - Usually variables aren't independent!
- Can use independence as a *modeling assumption*
 - Independence can be a simplifying assumption
 - Empirical* joint distributions: at best “close” to independent
 - What could we assume for {Weather, Traffic, Cavity}?

Example: Independence?

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$$P_2(T, W) = P(T)P(W)$$

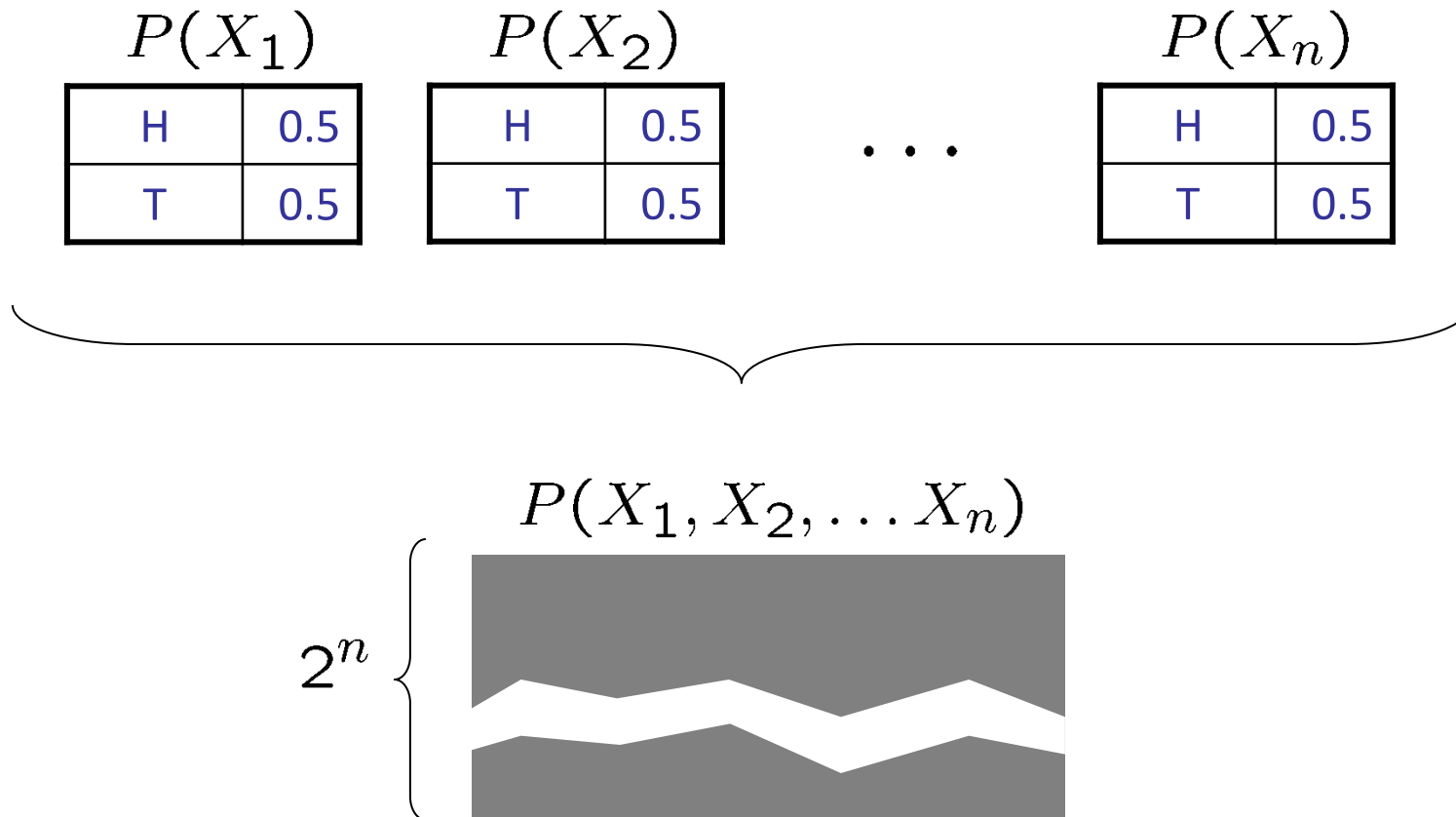
T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

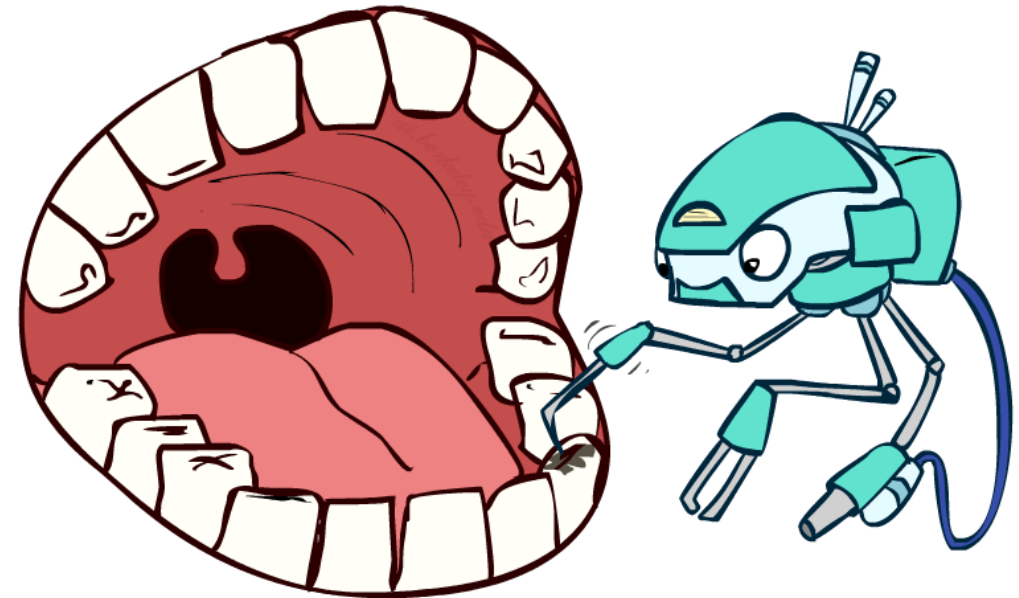
Example: Independence

- N fair, independent coin flips:



Conditional Independence

- $P(\text{Toothache}, \text{Cavity}, \text{Catch})$
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
 - $P(+\text{catch} \mid +\text{toothache}, +\text{cavity}) = P(+\text{catch} \mid +\text{cavity})$
- The same independence holds if I don't have a cavity:
 - $P(+\text{catch} \mid +\text{toothache}, -\text{cavity}) = P(+\text{catch} \mid -\text{cavity})$
- *Catch is conditionally independent of Toothache given Cavity:*
 - $P(\text{Catch} \mid \text{Toothache}, \text{Cavity}) = P(\text{Catch} \mid \text{Cavity})$
- **Equivalent statements:**
 - $P(\text{Toothache} \mid \text{Catch}, \text{Cavity}) = P(\text{Toothache} \mid \text{Cavity})$
 - $P(\text{Toothache}, \text{Catch} \mid \text{Cavity}) = P(\text{Toothache} \mid \text{Cavity}) P(\text{Catch} \mid \text{Cavity})$
 - One can be derived from the other easily



Conditional Independence

- Unconditional (absolute) independence very rare
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z

$$X \perp\!\!\!\perp Y | Z$$

if and only if:

$$\forall x, y, z : P(x, y | z) = P(x | z)P(y | z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x | z, y) = P(x | z)$$

Conditional Independence

- What about this domain:

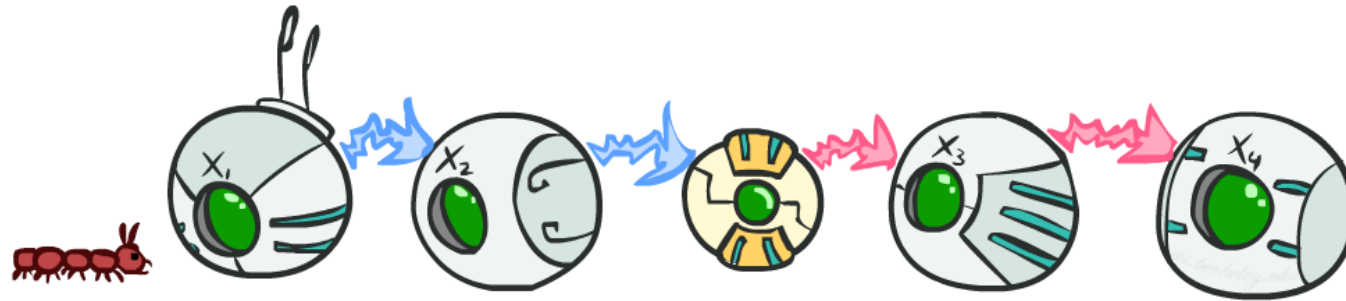
- Traffic
- Umbrella
- Raining

- What about this domain:

- Fire
- Smoke
- Alarm

Probability Recap

- Conditional probability $P(x|y) = \frac{P(x, y)}{P(y)}$
- Product rule $P(x, y) = P(x|y)P(y)$
- Chain rule
$$\begin{aligned} P(X_1, X_2, \dots, X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots \\ &= \prod_{i=1}^n P(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $\forall x, y : P(x, y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if: $X \perp\!\!\!\perp Y | Z$
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$



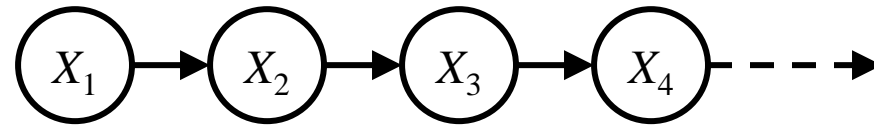
Markov Models

Reasoning over Time or Space

- Often, we want to **reason about a sequence** of observations
- Need to introduce time (or space) into our models

Markov Models

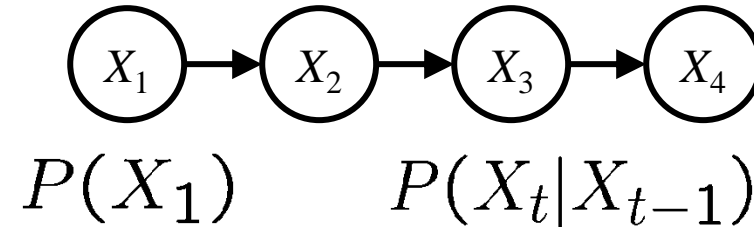
- Value of X at a given time is called the **state**



$$P(X_1) \quad P(X_t|X_{t-1})$$

- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity** assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

Joint Distribution of a Markov Model



- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

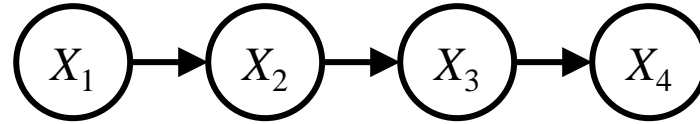
- More generally:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

Chain Rule and Markov Models



- From the chain rule, every joint distribution over X_1, X_2, X_3, X_4 can be written as:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

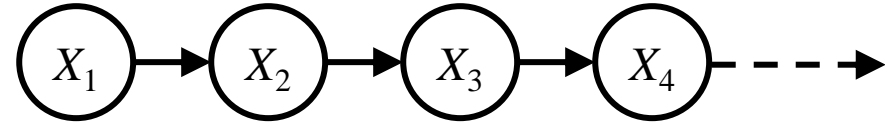
- Assuming that

$$X_3 \perp\!\!\!\perp X_1 \mid X_2 \quad \text{and} \quad X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

results in the expression posited on the previous slide:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

Chain Rule and Markov Models



- From the chain rule, every joint distribution over X_1, X_2, \dots, X_T can be written as:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_1, X_2, \dots, X_{t-1})$$

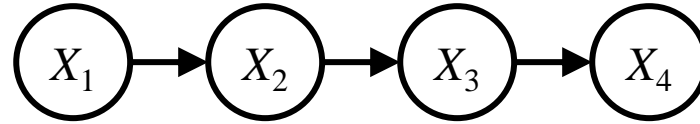
- Assuming that for all t :

$$X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$$

gives us the expression posited on the earlier slide:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1})$$

Implied Conditional Independencies



- We assumed: $X_3 \perp\!\!\!\perp X_1 \mid X_2$ and $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

- Do we also have $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$?

- Yes!

- Proof:

$$\begin{aligned} P(X_1 \mid X_2, X_3, X_4) &= \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)} \\ &= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)} \\ &= \frac{P(X_1, X_2)}{P(X_2)} \\ &= P(X_1 \mid X_2) \end{aligned}$$

Markov Models Recap

- Explicit assumption for all t : $X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

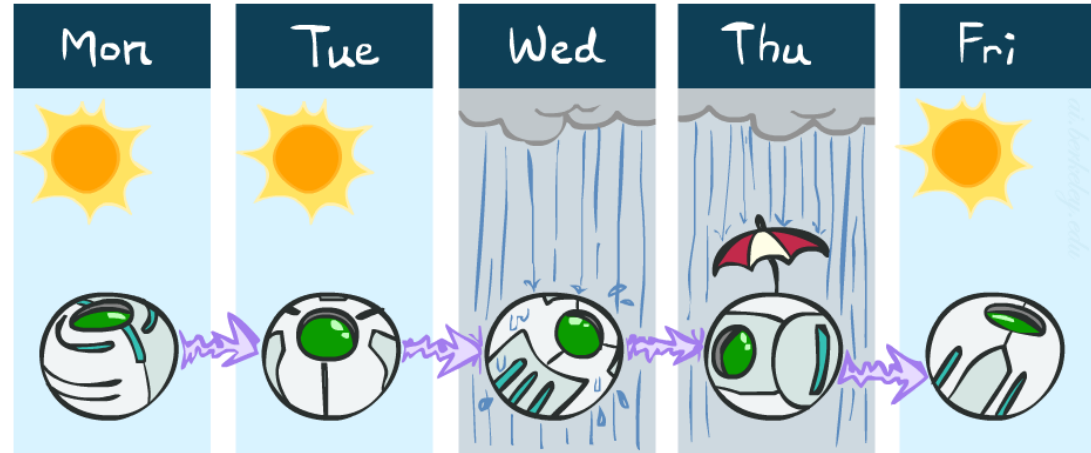
$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Implied conditional independencies: (try to prove this!)
 - Past variables independent of future variables given the present
i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all t

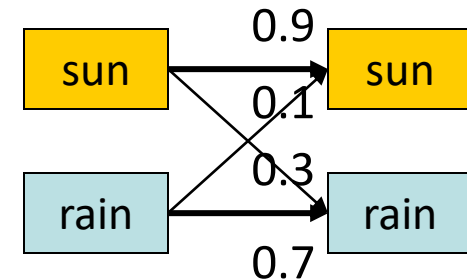
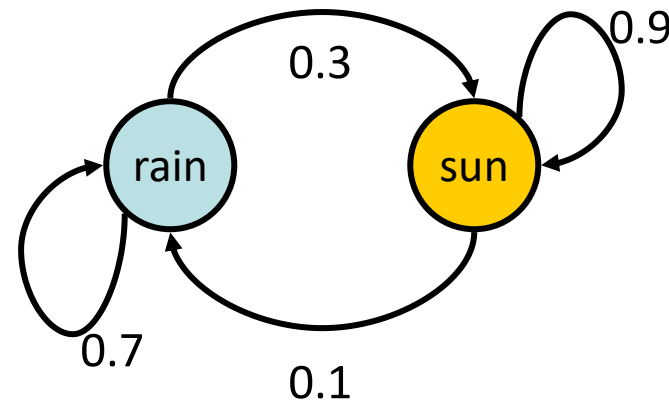
Example Markov Chain: Weather

- States: $X = \{\text{rain, sun}\}$
- Initial distribution: 1.0 sun
- CPT $P(X_t | X_{t-1})$:

X_{t-1}	X_t	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7



Two new ways of representing the same CPT



- P(Wednesday is Sunny)

- S S S: $0.9 * 0.9 = 0.81$

- S S R: $0.9 * 0.1 = 0.09$

- S R S: $0.1 * 0.3 = 0.03$

- S R R: $0.1 * 0.7 = 0.07$

- P(Wednesday is Sunny) =
0.84

- P(Saturday is Sunny)

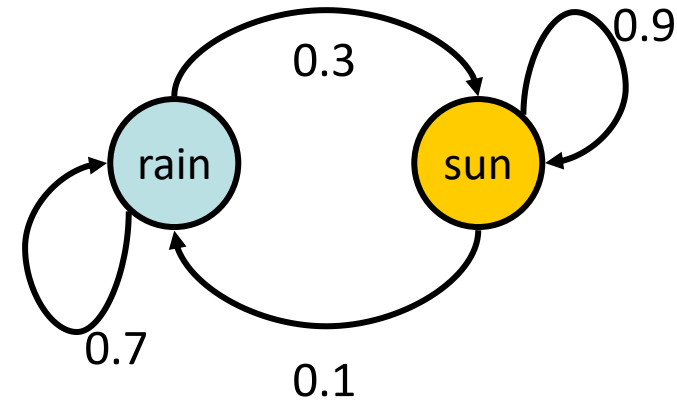
- S S S S S S

- S S S S R S

- S S S R S S

Example Markov Chain: Weather

- Initial distribution: 1.0 sun



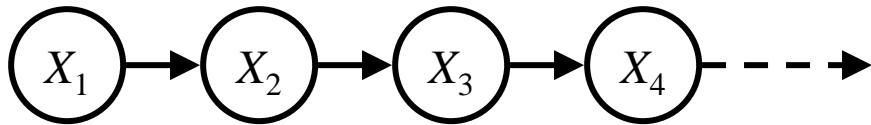
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

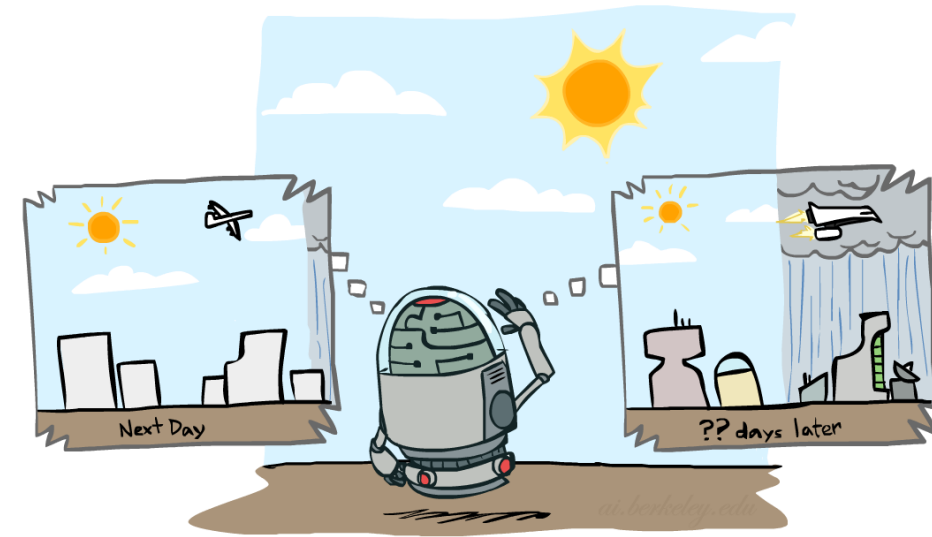
- Question: What's $P(X)$ on some day t ?



$P(x_1)$ = known

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

GWU *Forward simulation*



Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccccc}
 \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty)
 \end{array}$$

- From initial observation of rain

$$\begin{array}{ccccccc}
 \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty)
 \end{array}$$

- From yet another initial distribution $P(X_1)$:

$$\begin{array}{ccc}
 \left\langle \begin{array}{c} p \\ 1-p \end{array} \right\rangle & \dots & \longrightarrow \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & & P(X_\infty)
 \end{array}$$

Stationary Distributions

- For most chains:

- Influence of the initial distribution gets less and less over time.
- The distribution we end up in is independent of the initial distribution

- Stationary distribution:

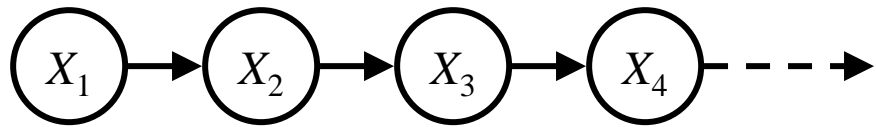
- The distribution we end up with is called the **stationary distribution** P_∞ of the chain
- It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$



Example: Stationary Distributions

- Question: What's $P(X)$ at time $t = \text{infinity}$?



$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

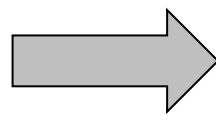
$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

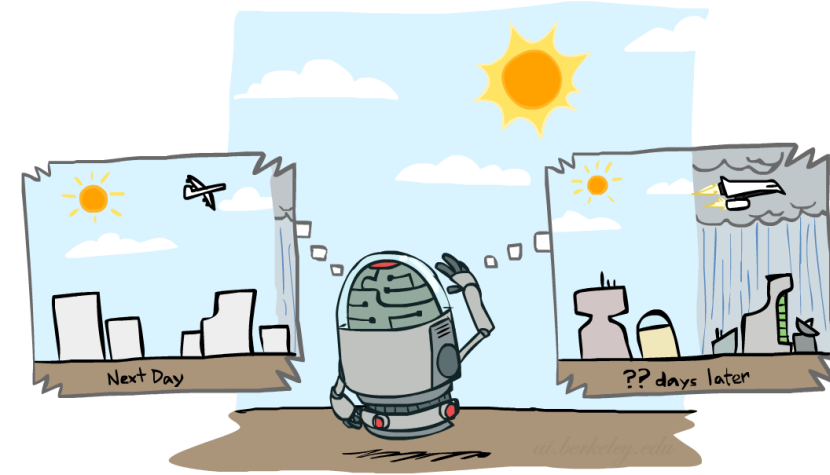
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

Also: $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$



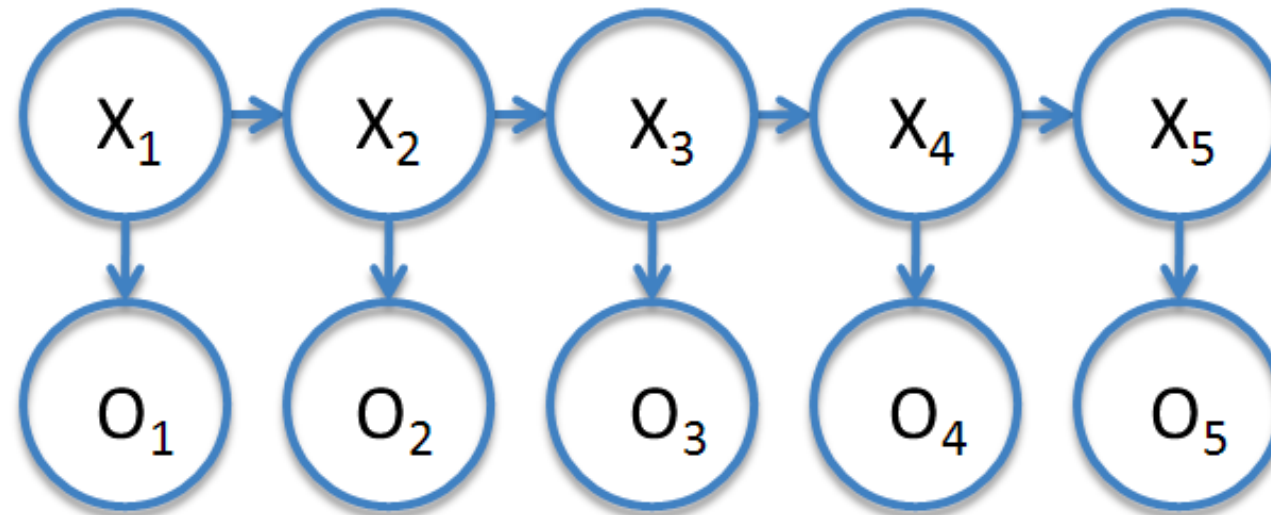
X_{t-1}	X_t	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Quiz: What is the Stationary Distribution?

From \ To →	Sun	Cloudy	Rain	
Sun	0.6	0.3	0.1	1
Cloudy	0.4	0.2	0.4	1
Rainy	0.1	0.4	0.5	1

CS 6511: Artificial Intelligence

Hidden Markov Models



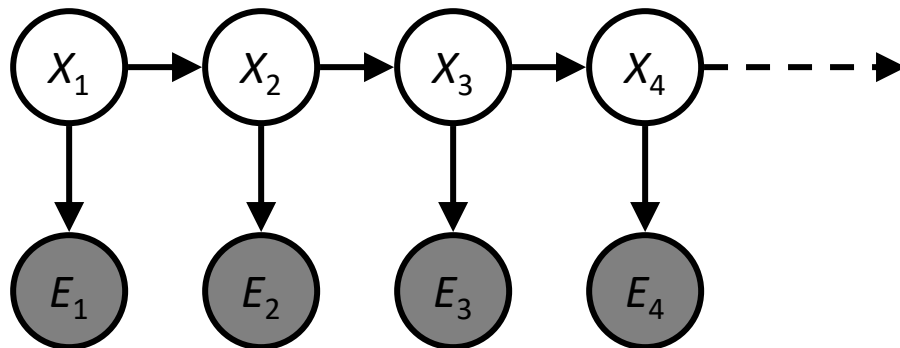
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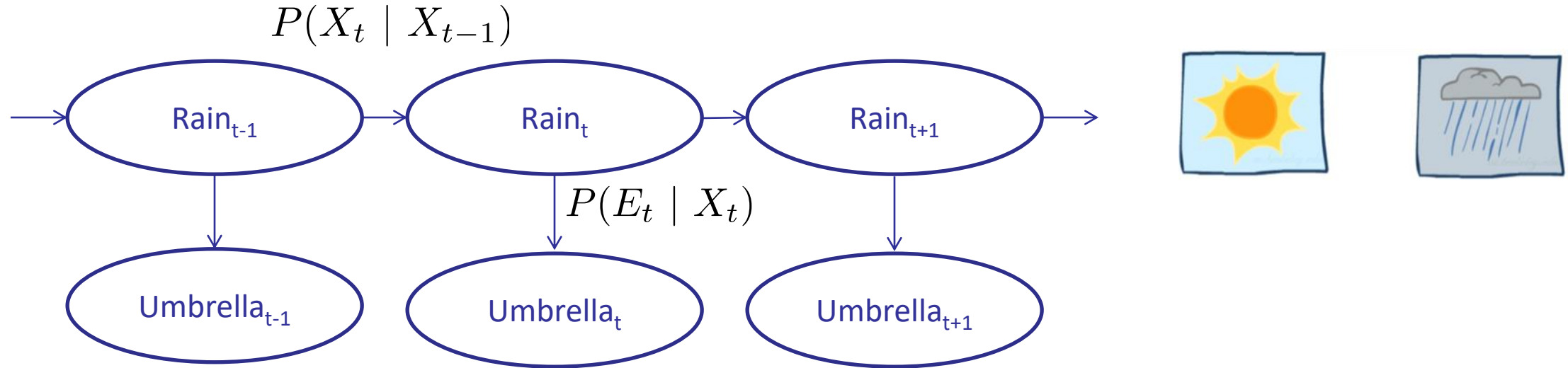
[Original version of these slides was created by Dan Klein and Pieter Abbeel for Intro to AI at UC Berkeley. <http://ai.berkeley.edu>]

Hidden Markov Models

- Markov chains not so useful for most agents
 - Need observations to update your beliefs
- Hidden Markov models (HMMs)
 - Underlying Markov chain over states X
 - You observe outputs (effects) at each time step



Example: Weather HMM



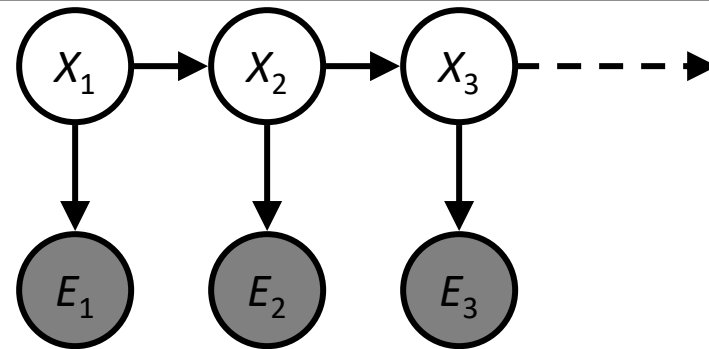
- An HMM is defined by:

- Initial distribution: $P(X_1)$
- Transitions: $P(X_t | X_{t-1})$
- Emissions: $P(E_t | X_t)$

R_t	R_{t+1}	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Joint Distribution of an HMM



- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

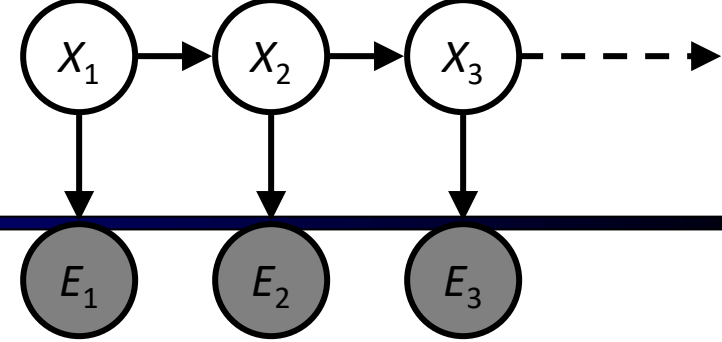
- More generally:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

Chain Rule and HMMs



- From the chain rule, every joint distribution over $X_1, E_1, X_2, E_2, X_3, E_3$ can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2) \\ P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

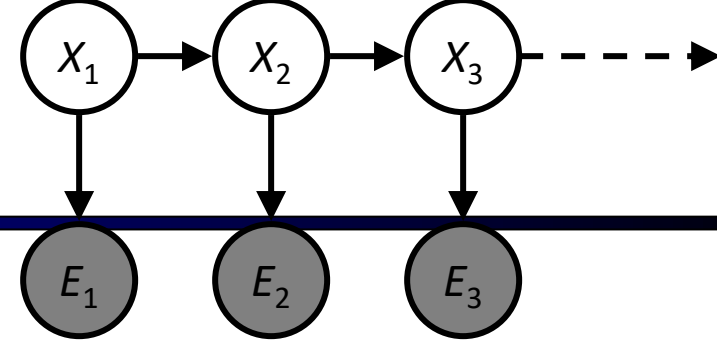
- Assuming that

$$X_2 \perp\!\!\!\perp E_1 \mid X_1, \quad E_2 \perp\!\!\!\perp X_1, E_1 \mid X_2, \quad X_3 \perp\!\!\!\perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp\!\!\!\perp X_1, E_1, X_2, E_2 \mid X_3$$

gives us the expression posited on the previous slide:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

Chain Rule and HMMs



- From the chain rule, every joint distribution over $X_1, E_1, \dots, X_T, E_T$ can be written as:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_1, E_1, \dots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \dots, X_{t-1}, E_{t-1}, X_t)$$

- Assuming that for all t :

- State independent of all past states and all past evidence given the previous state, i.e.:

$$X_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}$$

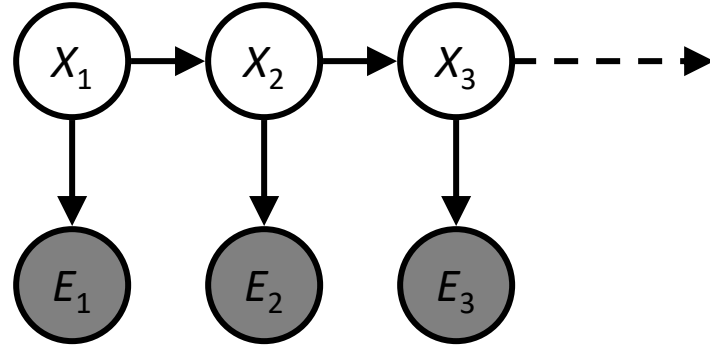
- Evidence is independent of all past states and all past evidence given the current state, i.e.:

$$E_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t$$

gives us the expression posited on the earlier slide:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

Implied Conditional Independencies



- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

- To prove them

- Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
- Approach 2: directly from the graph structure
 - Intuition: If path between U and V goes through W, then $U \perp\!\!\!\perp V \mid W$

Real HMM Examples

Speech recognition HMMs:

- Observations are acoustic signals (continuous valued)
- States are specific positions in specific words (so, tens of thousands)

Machine translation HMMs:

- Observations are words (tens of thousands)
- States are translation options

Robot tracking:

- Observations are range readings (continuous)
- States are positions on a map (continuous)

Filtering / Monitoring



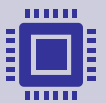
Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t | e_1, \dots, e_t)$ (the belief state) over time



We start with $B_1(X)$ in an initial setting, usually uniform

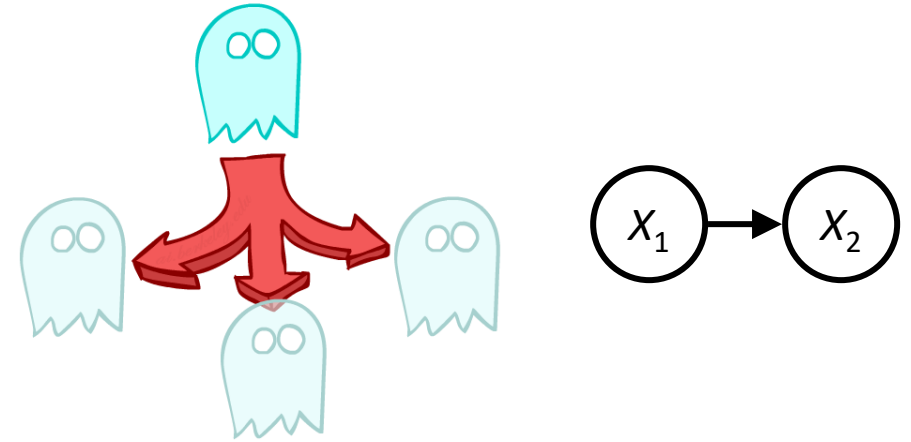
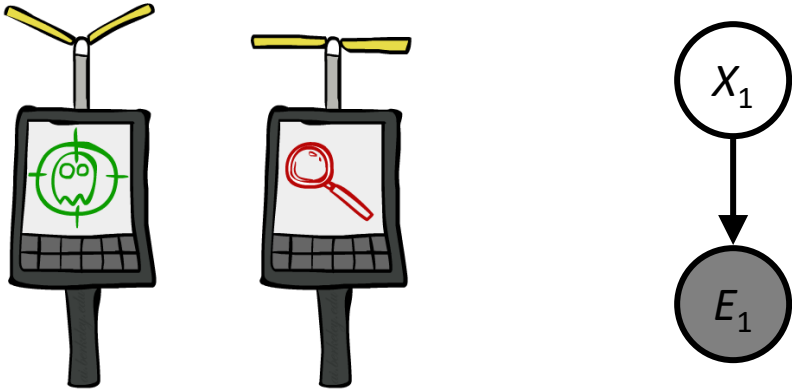


As time passes, or we get observations, we update $B(X)$



The Kalman filter was invented in the 60s and first implemented as a method of trajectory estimation for the Apollo program

Inference: Base Cases



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$

$$P(X_2)$$

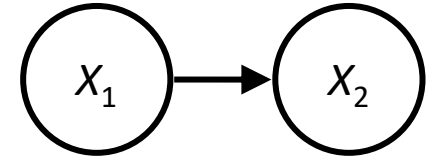
$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$

- Then, after one time step passes:



$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
 - With the “B” notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Summary

- Now you know all about Markov Models!
- HMMs are models in which we try to guess the hidden underlying state of the Markov Model by observing the emissions.
 - Forward algorithm can calculate the probability of a state given the sequence of emitted values.
 - Viterbi is a Dynamic Programming algorithm that finds the most likely explanation (To cover next week)